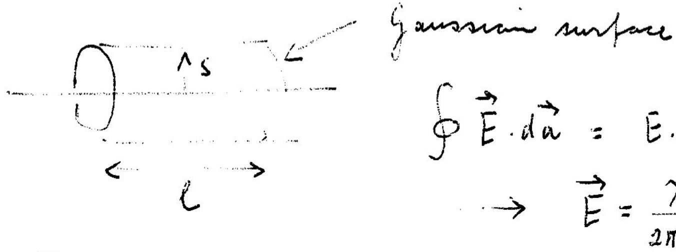
1) a)



3 pts

$$\oint \vec{E}\cdot d\vec{a} = E\cdot 2\pi s\, \ell = \frac{1}{\epsilon_0} Q_{enc} = \frac{1}{\epsilon_0}\ell\lambda$$

$$\rightarrow \quad \vec{E} = \frac{\lambda}{2\pi\epsilon_0 s}\,\hat{s}.$$

b) The charge extends to ∞, so we cannot choose the reference point at ∞.

Set it at $s = a$, then $V(s) = -\int_a^s \frac{\lambda}{2\pi\epsilon_0 \bar{s}}\, d\bar{s} = -\frac{1}{4\pi\epsilon_0} 2\lambda \ln\left(\frac{s}{a}\right)$

3 pts

$$\vec{\nabla} V = -\frac{\lambda}{2\pi\epsilon_0}\frac{\partial}{\partial s}\left[\ln\left(\frac{s}{a}\right)\right]\hat{s} = -\frac{\lambda}{2\pi\epsilon_0}\frac{1}{s}\hat{s} = -\vec{E} \quad \square.$$

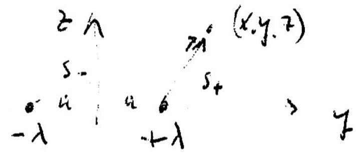
$a = \infty$ and $a = 0$ are no good!

c) Potential of $+\lambda$: $V_+ = -\frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{s_+}{a}\right)$, s_+ is distance from $+\lambda$

" " $-\lambda$: $V_- = +\frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{s_-}{a}\right)$, s_- " " " $-\lambda$

3 pts

$$\rightarrow \text{total } V = \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{s_-}{s_+}\right)$$



$$\left.\begin{aligned} s_+ &= \sqrt{(y-a)^2 + z^2} \\ s_- &= \sqrt{(y+a)^2 + z^2} \end{aligned}\right\} \rightarrow V(x,y,z) = \frac{\lambda}{4\pi\epsilon_0}\ln\left[\frac{(y+a)^2 + z^2}{(y-a)^2 + z^2}\right].$$

d) Equipotentials follow from:

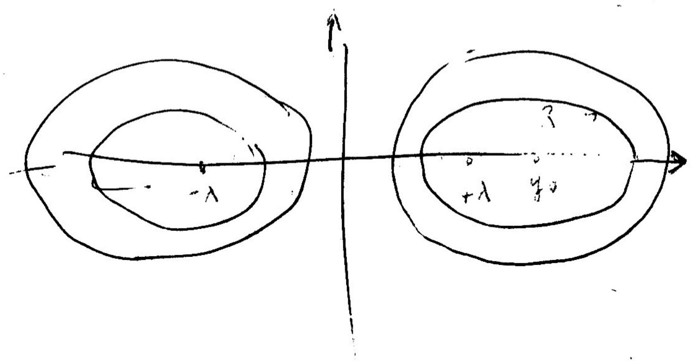
$$\frac{(y+a)^2 + z^2}{(y-a)^2 + z^2} = e^{4\pi\epsilon_0 V_0/\lambda} \equiv k = \text{constant}$$

$(y-y_0)^2 + z^2 = R^2$

3 pts

$$\rightarrow \quad y^2 + z^2 + a^2 - 2ay\left(\frac{k+1}{k-1}\right) = 0,$$ eqn of circle, center at $(y_0, 0)$ and radius R with $\left\{\begin{aligned} y_0 &= a\frac{k+1}{k-1} \\ R^2 &= a^2\frac{4k}{(k-1)^2} \end{aligned}\right.$

In terms of V_0: $\left\{\begin{aligned} y_0 &= a\coth\left(\frac{2\pi\epsilon_0 V_0}{\lambda}\right) \\ R &= a\,\text{csch}\left(\frac{2\pi\epsilon_0 V}{\lambda}\right) = \frac{a}{\sinh\left(\frac{2\pi\epsilon_0 V}{\lambda}\right)} \end{aligned}\right.$



a)  $\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc} \rightarrow B\, 2\pi s = \mu_0 I$

$\vec{B} \parallel \hat{\varphi}$ right-hand rule $\qquad \vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\varphi}$.

3 pts

b) from wire 1 : $\phi_1 = \int_{2d}^{3d} \frac{\mu_0 I d}{2\pi s}\, ds = \frac{\mu_0 I d}{2\pi} \ln\frac{3}{2}$, into the page

3 pts

from wire 2 : $\phi_2 = \int_{d}^{2d} \frac{\mu_0 I d}{2\pi s}\, ds = \frac{\mu_0 I d}{2\pi} \ln 2$, out of the page

→ total flux $\phi = \phi_2 - \phi_1 = \frac{\mu_0 I d}{2\pi} \ln\frac{4}{3}$.

c) $\mathcal{E} = -\frac{d\phi}{dt} = -\frac{\mu_0 d}{2\pi} \ln\frac{4}{3} \cdot \frac{dI}{dt}$.

4 pts

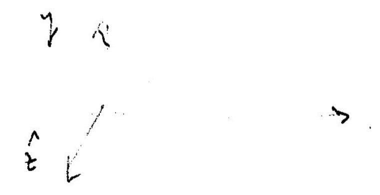
Lenz' law : Magn. field produced by the induced current tends to oppose the change of magn. flux, so this field will point into the page. Then (right-hand rule) the induced current is clockwise as seen from above.

d) B in a) was calculated using magnetostatics, although $\frac{dI}{dt} \neq 0$, this B was used on the right-hand side of Faraday's law. This is OK, when the rate $\frac{dI}{dt}$ is not too large and when not too far away from the source (otherwise retardation)

3 pts

In reality, there is also an $\vec{E}$ field due to the changing $\vec{B}$ and both $\vec{E}$ and $\vec{B}$ involve the retarded time $t - r/c$.

a) $\begin{cases} \vec{E} = E_0 \cos(kx - \omega t)\,\hat{y} \\ \vec{B} = \dfrac{E_0}{c} \cos(kx - \omega t)\,\hat{z} \end{cases}$ $\quad k = \dfrac{\omega}{c}$.



4 pnts

b) $\vec{S} = \dfrac{1}{\mu_0} \left(\vec{E} \times \vec{B}\right) = \dfrac{E_0^2}{\mu_0 c} \cos^2(kx - \omega t)\, \hat{y} \times \hat{z} \rightarrow \langle \vec{S} \rangle = \frac{1}{2} \varepsilon_0 c E_0^2\, \hat{x}$

2 pnts

$c = \dfrac{1}{\sqrt{\varepsilon_0 \mu_0}} \quad \mu_0 = \dfrac{1}{\varepsilon_0 c^2}$

c) $\begin{cases} E'_x = E'_z = 0 \\ E'_y = \gamma(E_y - v B_z) = \gamma E_0 \left(\cos(..) - \frac{v}{c} \cos(..) \right) = \alpha E_0 \cos(..) \end{cases}$

4 pnts

$\begin{cases} B'_x = B'_y = 0 \\ B'_z = \gamma\left(B_z - \frac{v}{c^2} E_y\right) = \gamma E_0 \left(\frac{1}{c} \cos(..) - \frac{v}{c^2} \cos(..)\right) = \alpha \frac{E_0}{c} \cos(..) \end{cases}$

where $\alpha = \gamma\left(1 - \frac{v}{c}\right) = \sqrt{\dfrac{1 - v/c}{1 + v/c}}$

Inverse Lorentz trafo: $\begin{cases} x = \gamma(\bar{x} + v\bar{t}) \\ t = \gamma(\bar{t} + \frac{v}{c}\bar{x}) \end{cases} \rightarrow kx - \omega t = \gamma\left[\left(k\beta - \frac{\omega v}{c^2}\right)\bar{x} - (\omega - kv)\bar{t}\right]$

$= \bar{k}\bar{x} - \bar{\omega}\bar{t}$

where $\bar{k} = \gamma\left(k - \frac{\omega v}{c^2}\right) = \gamma k (1 - v/c) = \alpha k$

and $\bar{\omega} = \gamma\omega\left(1 - v/c\right) = \alpha\omega$

$\rightarrow \begin{cases} \vec{E}'(x', y', z', t') = E'_0 \cos(k'x' - \omega' t')\,\hat{y} \\ \vec{B}'(x', y', z', t') = \dfrac{E'_0}{c} \cos(k'x' - \omega' t')\,\hat{z} \end{cases}$

where $E'_0 = \alpha E_0$, $k' = \alpha k$, $\omega' = \alpha\omega$.

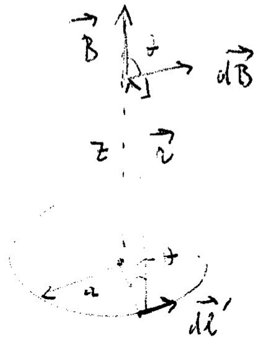
d) $\omega' = \omega \sqrt{\dfrac{1 - v/c}{1 + v/c}}$ Doppler shift for light, $\lambda' = \dfrac{2\pi}{k'} = \dfrac{2\pi}{\alpha k} = \dfrac{\lambda}{\alpha}$

3 pnt

$v' = \dfrac{\omega'}{2\pi} \lambda' = \dfrac{\omega}{\lambda} = c$, indep of inertial frame!

$I \sim E_0^2 \rightarrow \dfrac{I'}{I} = \dfrac{E_0'^2}{E_0^2} = \alpha^2 = \dfrac{1 - v/c}{1 + v/c}$

so amplitude, freq and intensity $\rightarrow 0$ as $v \rightarrow c$ (red-shifted) but still

a) 

3 pts

As we integrate $\vec{dl}'$, $\vec{dB}$ sweeps out a cone, horizontal components cancel, vertical components add up →

$$B(z) = \frac{\mu_0}{4\pi} I \int \frac{dl'}{r^2} \cos\theta$$

→ projects out vertical component

$$= \frac{\mu_0 I}{4\pi} \frac{\cos\theta}{r^2} 2\pi a = \frac{\mu_0 I}{2} \frac{a^2}{(a^2 + z^2)^{3/2}} .$$

b) From coil 1: $B_1(x) = \frac{\mu_0 I}{2} \frac{a^2}{\left(a^2 + (\frac{1}{2}b + x)^2\right)^{3/2}}$ to the right, $z = \frac{1}{2}b + x$

3 pts

From coil 2: $B_2(x) = \frac{\mu_0 I}{2} \frac{a^2}{\left(a^2 + (\frac{1}{2}b - x)^2\right)^{3/2}}$ also to the right. $z = \frac{1}{2}b - x$

$$B = B_1 + B_2 .$$

c) $a^2 + (\frac{1}{2}b \pm x)^2 = (a^2 + \frac{1}{4}b^2)\left[1 \pm \frac{bx}{a^2 + \frac{1}{4}b^2} + \frac{x^2}{a^2 + \frac{1}{4}b^2}\right]$ $c^2 \equiv a^2 + \frac{1}{4}b^2$.

3 pts

$\cancel{f(x) = \dots \pm bx}$ $g(y) = (1+y)^{-3/2} = g(0) + y g'(0) + \frac{1}{2} y^2 g''(0) + \dots$

$= 1 - \frac{3}{2} y + \frac{15}{8} y^2 + \dots$ $y = \frac{\pm bx + x^2}{c^2}$

$$\rightarrow f(x) = (c^2)^{-3/2} \left[1 + \frac{\pm bx + x^2}{c^2}\right]^{-3/2} = c^{-3} \left[1 \mp \frac{3b}{2c^2} x + \underbrace{\left(\frac{15b^2}{8c^4} - \frac{3}{2c^2}\right)}_{\equiv 0} x^2\right]$$

d) $\frac{5}{4} b^2 = c^2 = a^2 + \frac{1}{4} b^2$

↓

$\underline{a = b}$ then the field is homogeneous to order x^2.

3 pts

$$\rightarrow B = \frac{\mu_0 I}{2} \frac{a^2}{(a^2 + \frac{1}{4}b^2)^{3/2}} \cdot 2 = \frac{\mu_0 I}{2} \frac{a^2}{(\frac{5}{4}a^2)^{3/2}} \cdot 2 = \mu_0 I \frac{4^{3/2}}{5^{3/2} a} = \frac{8 \mu_0 I}{5^{3/2} a} \quad \Box$$